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Information Processing Models Generating Lognormally Distributed Reaction Times

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The lognormal distribution has been fitted successfully to empirical reaction times (RTs), yet mechanisms that might generate lognormally distributed (RTs are unknown. Thus, the lognormal has the status of an ad-hoc distribution in RT research. In this paper we describe mechanisms that could generate the lognormal distribution and show how specific RT models can be constructed within the framework of general mathematical properties of the lognormal. It is demonstrated that various conceptually different mechanisms could produce lognormally distributed RTs. © 1993 Academic Press, Inc.

INFORMATION PROCESSING MODELS GENERATING LOGNORMALLY DISTRIBUTED REACTION TIMES

The lognormal distribution sometimes provides an excellent fit to empirical RT distributions. This point was first noted by researchers making direct comparisons of observed and expected frequencies of responses in given RT ranges (e.g., Brée, 1975; Ratcliff &Murdock, 1976; Schlosberg & Heineman, 1950; Woodworth & Schlosberg, 1954). More recently, it has been reinforced by the finding that empirical hazard functions of RT can first increase and then decrease to zero, as does the hazard function of the lognormal (e.g., Green & Smith, 1982, Fig. 6, 50 ms signals). In certain experiments the obtained hazard functions do not match the shape predicted by the lognormal (e.g., Burbeck & Luce, 1982; Green & Smith, 1982, Fig. 6, 1 s signals), bu the deviations may be due to the practice of throwing out all RTs greater than a certain fixed upper cutoff (Ulrich & Miller, 1992).

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Despite its empirical success as a model of RT distributions, information processing mechanisms that could potentially generate lognormally distributed RTs are unknown, so the lognormal has the status of an ad-hoc distribution in RT-research (Luce, 1986). The purpose of this paper is to describe mechanisms that would generate lognormal RTs and hence provide a theoretical justification for the lognormal in RT-research. Interestingly, it turns out that lognormal RTs are easily generated by certain models that are conceptually very different from traditional serial RT models (Sternberg, 1969).

1. The Lognormal

The lognormal distribution is usually defined by saying that the time **T** follows a lognormal if the logarithm ln(**T**) of time **T** is normally distributed with mean $E[\ln(\mathbf{T})]=\mu$ and standard deviation $\mathrm{SD}[\ln(\mathbf{T})]=\sigma$. **T** is necessarily a positive random variable, and its mean and variance are $\exp[\mu+\sigma^2/2]$ and $[\exp(2\mu+\sigma^2)]\times[\exp(\sigma^2)-1]$, respectively. The lognormal distribution is usually considered as a possible model in statistics whenever a random variable with a positive skewness is needed (e.g., Crow & Shimizu, 1988).

The PDF of the lognormal is unimodal and for $t>0$ given by

$$f(t)=\frac{1}{\sqrt{2\pi}\sigma t}\exp\left[-\frac{(\ln t-\mu)^2}{2\sigma^2}\right]. \tag{1}$$

The parameters $\sigma>0$ and μ are the shape parameter and scale parameter of the lognormal, respectively. Skewness increases with σ.

In addition, **T** is usually said to follow a lognormal distribution if $\ln(\mathbf{T}-t_0)$ has a normal distribution, where t_0 is any positive constant. This allows **T** to have the lognormal shape but to be displaced by t_0 units to the right on the time axis. In this case, **T** has a mean t_0 units larger and the same variance.[1]

The lognormal distribution arises in many branches of biology (Koch, 1966, 1969) and economics (Lawrence, 1988), and it has also been applied to phenomena in several other areas (see reviews in Crow & Shimizu, 1988), such as psychophysics (Ennis & Mullen, 1992) and psycholinguistics (Howes, 1971). Thus, there has been some previous consideration of the types of processes that give rise to this distribution. Two processes have been of particular significance in the statistical literature (compare Shimizu & Crow, 1988). The first process arises as the transform of an underlying fluctuation that follows a normal distribution, while the second process yields the lognormal distribution directly rather than as a transform of a normal. As we will show, these processes can also arise in the domain of RT modeling.

[1] It is rather difficult to estimate the three parameters of the (potentially) shifted lognormal distribution. The most promising approach appears to be a modified method of moments proposed by Cohen and Whitten (1980) and elaborated by Cohen, Whitten, and Ding (1985) and Cohen (1988).

Assume that $\mathbf{X}$ is normally distributed with parameters μ and σ and let $\mathbf{T}$ be an observable latency that is related to $\mathbf{X}$ as

$$\mathbf{T} = \exp(\mathbf{X}). \tag{2}$$

Then $\mathbf{T}$ follows a lognormal with parameters μ and σ. This is easily seen, because $\mathbf{T} = \exp[\ln(\mathbf{T})]$ and hence $\ln(\mathbf{T}) = \mathbf{X}$.

This exponential transform property can be generalized to

$$\mathbf{T} = a \cdot \exp(c\mathbf{X} + d) + t_0, \tag{3}$$

where a, t_0, c, and d are any positive constants The distribution of $\mathbf{T}$ given by (3) is a lognormal displayed by t_0 units to the right on the time axis, with mean

$$E[\mathbf{T}] = a \cdot \exp\left[c\mu + d + \frac{c^2 \cdot \sigma^2}{2}\right] + t_0 \tag{4}$$

and variance

$$\mathrm{Var}[\mathbf{T}] = a^2 \cdot \exp[2(c\mu + d) + c^2\sigma^2] \times [\exp(c^2\sigma^2) - 1]. \tag{5}$$

The lognormal may also arise directly (i.e., without the existence of an underlying normal variate) when one random variable is formed as the product of a number of other ones. Let

$$\mathbf{X}_n = \prod_{i=1}^{n} \mathbf{Y}_i, \tag{6}$$

where $\{\mathbf{Y}_1, ..., \mathbf{Y}_n\}$ is a sequence of independent, identically distributed random variables. Then

$$\ln(\mathbf{X}_n) = \sum_{i=1}^{n} \ln(\mathbf{Y}_i). \tag{7}$$

According to the central limit theorem, the sum on the right side of (7) converges to a normal distribution—and thus $\mathbf{X}_n$ converges to a lognormal distribution—rapidly as n increases, under fairly general conditions (cf., Feller, 1971, pp. 262–264; Wolfson, 1985).

2. Specific RT Models

In this section we consider specific RT models of the forms outlined in the preceding section. The first model has a single stage of activation growth, and the lognormal emerges from the underlying normal distribution of a response criterion against which activation is compared (cf. Grice, 1968; Schlosberg & Heineman, 1950). The second model is based on the transmission of partial activation across

multiple stages, and the lognormal emerges because RT is determined by the product of many randomly varying rates of activation growth. One might object that the assumptions underlying these models are just as ad-hoc as the assumption of lognormally distributed RTs, but this objection misses the point of our enterprise. Our goal is to identify underlying mechanisms that can generate lognormal distributions of RTs, and further research will be needed to determine whether these mechanisms are actually used in particular experimental paradigms.

Logarithmic Activation Growth

Suppose that a stimulus requiring a speeded response is presented at time $t=0$ and that response activation $A(t)$ begins to accumulate as a logarithmic function of t,

$$A(t)=k\cdot\ln(t). \tag{8}$$

The response is triggered when $A(t)$ reaches a criterion $\mathbf{C}>0$, and $\mathbf{C}$ is subject to random trial-to-trial fluctuation.

Let $\mathbf{T}$ be the point in time when activation A reaches criterion $\mathbf{C}$. Therefore, we have

$$A(\mathbf{T})=\mathbf{C} \tag{9}$$

$$k\cdot\ln(\mathbf{T})=\mathbf{C} \tag{10}$$

and thus

$$\mathbf{T}=\exp(\mathbf{C}/k). \tag{11}$$

According to (11), latency $\mathbf{T}$ will be lognormally distributed if $\mathbf{C}$ is normally distributed.[2] Substituting $1/k$ for c and letting $a=1$ and $d=0$ in (4) and (5) yields, respectively,

$$E[\mathbf{T}]=\exp\left[\frac{\mu}{k}+\frac{\sigma^2}{2k^2}\right] \tag{12}$$

and variance

$$\mathrm{Var}[\mathbf{T}]=\exp\left(\frac{2\mu}{k}+\frac{\sigma^2}{k^2}\right)\times[\exp(\sigma^2/k^2)-1]. \tag{13}$$

[2] In model (11) and the related model (15), it is assumed that T represents the decision latency (see Luce, 1986, Chap. 3), which is an unobservable component of the total RT. Therefore, both models neglect motor components of RT. However, there is some empirical (Wing & Kristofferson, 1973; Ulrich & Stapf, 1984) and theoretical (Ulrich & Wing, 1991) evidence that the motor component contributes little to the total variance of RT, so it is not unreasonable to neglect this component for the purpose of model building. Thus, we assume constant motor processing times within these models.

The logarithmic activation growth is conceptually similar to the associative strength model recently discussed by Grice, Canham and Boroughs (1984). According to their model, associative strength A builds exponentially over time, according to

$$A(t) = a - m \cdot \exp(-kt). \tag{14}$$

As before, it is assumed that a response is triggered as soon as $A(t)$ crosses a normally distributed criterion $\mathbf{C} > 0$. Thus, the associative strength model implies

$$\mathbf{T} = -\frac{1}{k} \ln\left(\frac{a-\mathbf{C}}{m}\right). \tag{15}$$

Thus it can be seen that the associative strength model does not imply lognormally distributed latencies. The distribution resulting from Grice's model and its fit to empirical RT distributions are discussed by Luce (1986, pp. 150–151), who reports a bad fit.[3]

Partial-Output Models

Starting with the influential paper of McClelland (1979), many theorists have considered specific models in which RT is determined by a series of processes cascading activation from an input level to an output level, usually passing through a number of intervening processing levels along the way. Contrary to traditional serial processing models (see Miller, 1988, for review and further analyses), these partial output models allow a given processing level to start transmitting output (i.e., activation) before it has finished processing (i.e., reached its asymptotic activation level).

[3] Actually, it appears to us that Grice's model is ill-defined because it predicts a probability larger then zero that no response will occur on a given trial. To see this, we note that

$$\Pr\{\mathbf{T} > t\} = \Pr\{A(t) < \mathbf{C}\} \tag{16}$$

$$= \Pr\{a - m \cdot \exp(-kt) < \mathbf{C}\}. \tag{17}$$

For $t \to \infty$ and $a < \infty$, we have

$$\Pr\{\mathbf{T} > t\} = \Pr\{a < \mathbf{C}\} \tag{18}$$

$$= 1 - \Pr\{\mathbf{C} \leqslant a\} \tag{19}$$

$$> 0, \tag{20}$$

since $\mathbf{C}$ is a normal random variable. This means that there is a non-zero chance that a response will never occur. Whether or not this probability is negligibly small depends on the mean and variance of $\mathbf{C}$ and the asymptote a.

A similar objection has been made by Ashby (1982) to the cascade model of McClelland (1979). For this model, Ashby derived the density of RT conditional on a response occurring and based further predictions on this density. Unfortunately, we cannot tell from the text of Luce whether such an adjustment was also used in fitting Grice's model to empirical RT data.

In a recent paper, Schweickert (1989) described a general form of such models having two processes, say x and y, in series. The output $g(t)$ of process x increases monotonically with time t and is continuously fed into process y. Similarly, the output $f[g(t)]$ of y is monotonically related to its input $g(t)$ at time t. Hence the output of y is the composition of two monotonic increasing functions, namely $g(t)$ and $f(t)$.

The description of partial output models as a composition can be extended to a chain of n successive processes or units in which the output from unit $(i-1)$ at time t serves as the input to unit i. The output $O_n(t)$ of the last unit, n, in the chain is given by the composition

$$O_n(t) = g_n(\cdots g_3(g_2(g_1(t)))\cdots), \tag{21}$$

where $g_i(t)$ is a monotonic increasing function denoting the output of unit i $(i=1, ..., n)$ at time t. It is obvious that $O_n(t)$ will be mathematically intractable without specific assumptions about the shape of $g_i(t)$. The lognormal arises from the assumption that the input/output relationship of unit $i>1$ is a power function,

$$g_i(t) = \mathbf{A}_i \cdot [g_{i-1}(t)]^{b_i}, \tag{22}$$

where the slope $\mathbf{A}_i$ is a positive random variable and $b_i>0$ is a positive constant. Thus, the output g_i is a negatively accelerated function of the input for $0<b_i<1$, a linear function for $b_i=1$, and a positively accelerated function for $b_i>1$. The first unit, $i=1$, in the chain is called the receptor unit, and its power function is given by

$$g_1(t) = \begin{cases} \mathbf{A}_1 \cdot (t-t_0)^{b_1} & \text{if} \quad t>t_0 \\ 0 & \text{otherwise,} \end{cases} \tag{23}$$

where t_0 is a startup delay in the transmission of activation. Finally, it is assumed that the response to a signal at $t=0$ is triggered as soon as the output $O_n(t)$ reaches a constant criterion value c. (We later relax this restriction and allow c to vary.) According to this model, random trial-to-trial fluctuations of latency $\mathbf{T}$ are solely attributable to variation in the slopes $\mathbf{A}_1, ..., \mathbf{A}_n$.

To derive the distribution of $\mathbf{T}$, we apply (23) and (22) to (21). For $t>t_0$, this yields

$$O_n(t) = \mathbf{A}_n \cdot \mathbf{A}_{n-1}^{b_n} \cdot \mathbf{A}_{n-2}^{n_n \times b_{n-1}} \cdots \mathbf{A}_1^{b_n \times \cdots \times b_2} \cdot (t-t_0)^{b_1 \times \cdots \times b_n}. \tag{24}$$

Let $k_i \equiv \prod_{j=i+1}^{n} b_j$ for $i=0, ..., n-1$ and $k_i \equiv 1$ for $i=n$, which enables us to write (24) more compactly as

$$O_n(t) = (t-t_0)^{k_0} \times \prod_{i=1}^{n} \mathbf{A}_i^{k_i}. \tag{25}$$

Since a response is triggered when event $\{O_n(\mathbf{T}) = c\}$ occurs, one obtains from (25)

$$c = (\mathbf{T} - t_0)^{k_0} \times \prod_{i=1}^{n} \mathbf{A}_i^{k_i}. \tag{26}$$

Taking the natural logarithm on both sides gives

$$\ln(c) = k_0 \cdot \ln(\mathbf{T} - t_0) + \sum_{i=1}^{n} [k_i \cdot \ln(\mathbf{A}_i)]. \tag{27}$$

Rearranging yields

$$\ln(\mathbf{T} - t_0) = \frac{\ln(c) - \sum_{i=1}^{n} [k_i \cdot \ln(\mathbf{A}_i)]}{k_0}. \tag{28}$$

Taking the antilog on both sides and rearranging gives

$$\mathbf{T} = \exp\left\{\frac{\ln(c) - \sum_{i=1}^{n} [k_i \cdot \ln(\mathbf{A}_i)]}{k_0}\right\} + t_0. \tag{29}$$

If we abbreviate $\mathbf{X}_0 \equiv \ln(c)/k_0$ and $\mathbf{X}_i \equiv -k_i \cdot \ln(\mathbf{A}_i)/k_0$ then we arrive at

$$\mathbf{T} = \exp\left(\sum_{i=0}^{n} \mathbf{X}_i\right) + t_0. \tag{30}$$

Under relatively general conditions of the central limit theorem, the sum in (30) is expected to approach a normal distribution and thus **T** should approximately follow a lognormal distribution shifted by t_0. Note that our final result even allows the criterion c to vary randomly from trial to trial, although it must remain constant over time within a trial.[4]

A series of simulations was conducted in order to examine the rate at which the sum in (30) converged to the normal distribution. The series had a factorial structure, with each simulation defined by a combination of:

[4] Since **C** and $\mathbf{A}_i$ must be positive random variables, a positive skewness of their distributions appears to be most likely. If a positively skewed distribution is log-transformed then it usually becomes more symmetrical and hence closer to the normal distribution. Therefore, in this case, **T** is expected to approximate a lognormal distribution rapidly. Ironically, **T** would exactly follow a lognormal distribution if **C** and $\mathbf{A}_1, ..., \mathbf{A}_n$ were lognormally distributed, because the logarithm of a lognormally distributed random variable yields a normal random variable and the sum of normal variables is again normally distributed; thus the exponentiated term in (30) would contain only normal random variables and thus **T** would be a (shifted) lognormal variable.

Intuitively, it seems that the distribution of the sum $\sum_{i=0}^{n} \mathbf{X}_i$ should be close to a normal distribution if the individual variances $\mathrm{Var}[\mathbf{X}_i]$ $(i = 0, ..., n)$ do not differ too much. This ideal condition, however, imposes a certain restriction on the b_i's; namely, $k_i \approx 1$ for $i = 0, ..., n$. This restriction, for example, would be fullfilled if positively and negatively accelerated power functions would alternate in the processing chain. However, the simulations discussed next clearly demonstrate that this alternation-restriction is not necessary in order to obtain a close approximation to the normal distribution for $\sum_{i=0}^{n} \mathbf{X}_i$.

- the number of units n, which equalled 1, 2, 3, 4, 6, 8, 10, 15, or 25;
- the value of b_i—equal for all units—which was 0.5, 0.75, 1.0, 1.5, or 2.0; and
- the probability distribution for $\mathbf{A}_i$, which was either
  - — uniform ranging from 0.5 to 1.5,
  - — normal with a mean of 1.0 and a standard deviation of 0.15,
  - — exponential with a mean of 1.0, or
  - — gamma: the sum of two exponentials each with mean 0.5.

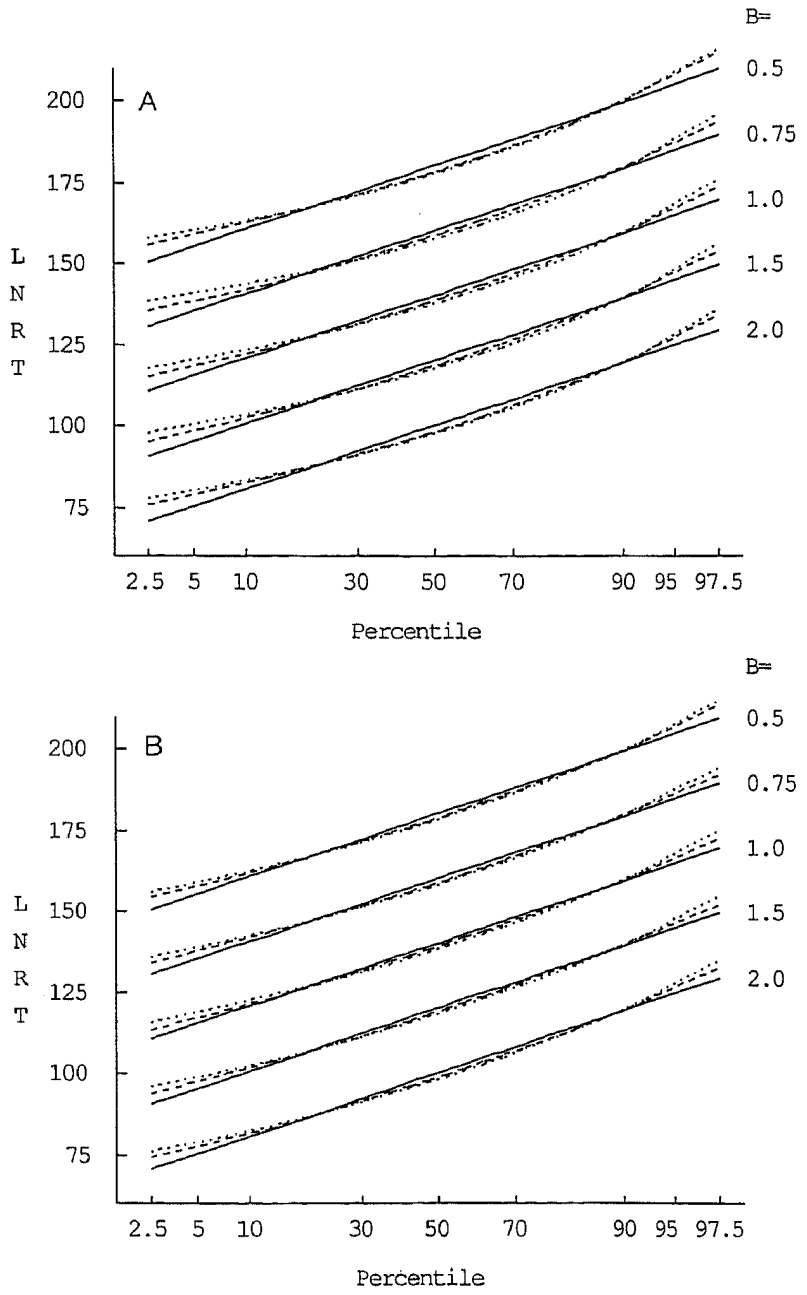

FIG. 1. Simulated RT distributions obtained from (30), showing rate of convergence to lognormal. Simulations were conducted separately for exponential, gamma, normal, and uniform distributions of the A_i's (Panels A through D, respectively) and for the five values of the b_i's indicated by the curve parameters. The dotted and dashed lines show simulation results for one- and four-unit models (i.e., $n=1$ and $n=4$), respectively, and the solid lines show exact normal distributions for comparison.

In each simulation, 10,000 RT values were generated. To generate an RT, values for $\mathbf{A}_i$ were first randomly generated independently for each unit, and then (28) was solved for ln(T). Without loss of generality, t_0 was set to zero. The 10,000 ln(T) values generated in each simulation were used to estimate the values of the ln(T) distribution at 20 percentiles ranging from 2.5 to 97.5 in steps of 5.0.

Figure 1 shows representative results from the simulations. The vertical axis shows the obtained percentile values for ln(T) in each simulation. Since the units of this axis are arbitrary, the ln(T) values have been adjusted to different means for the different values of b_i so that the lines would not overlap on the graphs. Percentiles are shown on the horizontal axis, which has been scaled so that a

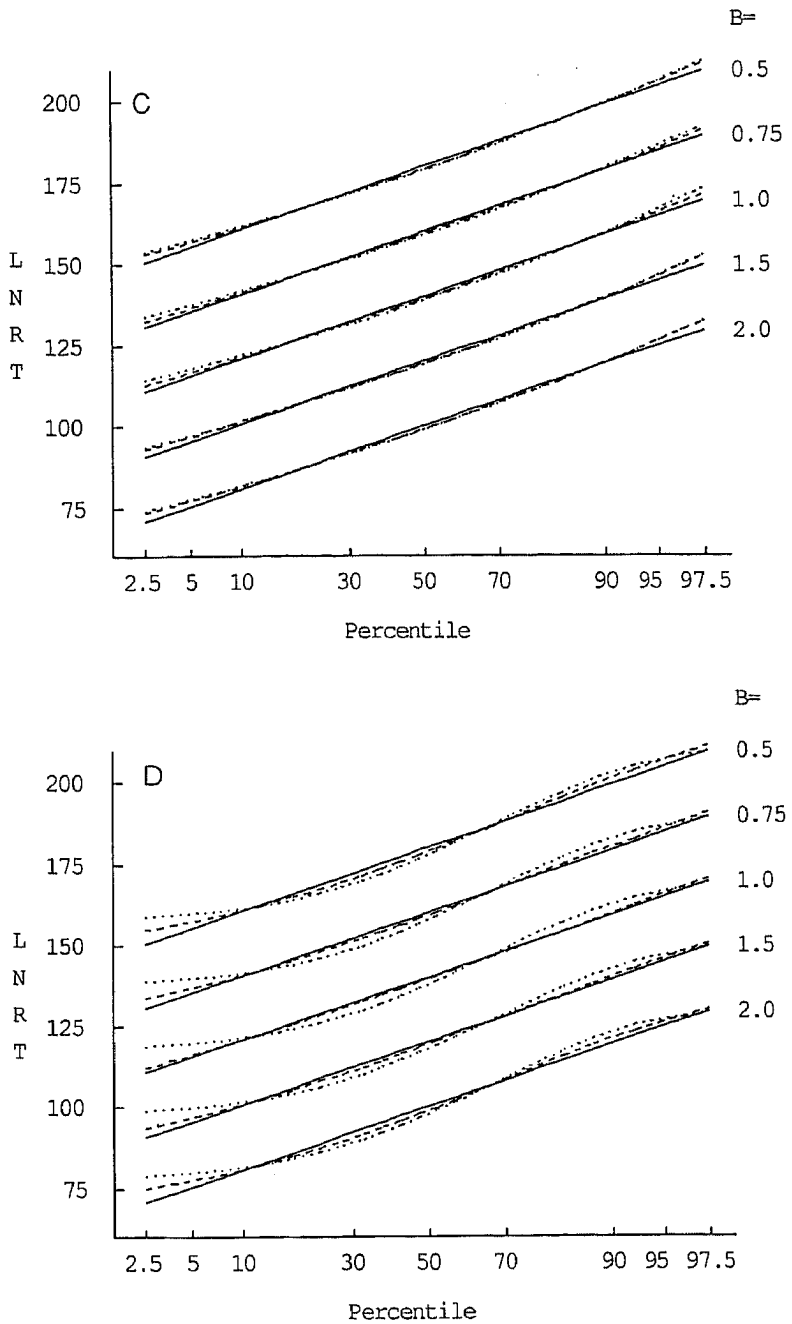

FIG. 1—*Continued*

normal distribution appears as a straight line. Thus, the results approach an exact lognormal to the extent that the simulated data values fall on or near a straight line.

The graphs show that the simulated values of ln(T) approximate the normal distribution rather well. The simulated values are shown for the cases of $n = 1$ (dotted lines) and $n = 4$ (dashed lines), and exact normal distributions are shown for comparison purposes (straight lines). It is clear that the results with $n = 4$ give a very good approximation to a normal distribution. Interestingly, the small departures from normality were *not* generally eliminated by increasing the number of units up to 25.

In a second series of simulations, we measured the convergence of the sum in (30) to a normal by estimating the power of actual experiments with 50, 100, 200, 400, or 800 observations to detect the small departures from normality shown in Fig. 1. This series was restricted to parameter combinations with $n = 1$, 4, 10, or 25 and $b_i = 0.5$, 1.0, or 2.0. For each combination of parameters, 1000 experiments were simulated. For each experiment, the appropriate numbers of observations was randomly generated, and Geary's test of normality (cf., D'Agostino, 1970) was applied to see whether the hypothesis of normality could be rejected for the obtained distribution. Thus, the percentage of samples for which normality could be rejected was the index of goodness of fit to the normal distribution. The results give a more precise picture of the size of the deviations from normality. For example, in experiments with 400 observations, the percentage of null hypotheses rejected was 55, 31, 30, and 28 for $n = 1$, 4, 10, and 25, respectively. The pattern was similar for other numbers of observations, although of course the overall percentage of rejections was larger with more observations (e.g., for $n = 1$, the percentage of rejections was 22, 34, 45, and 68 in experiments with 50, 100, 200, and 800 observations, respectively). Convergence to normality was essentially perfect with b_i's equal to 1.0, in the sense that the percentage of rejections was 5% for $n = 25$.

A third set of simulations was conducted to examine the rate of convergence of (30) when the b_i's varied randomly. Specifically, simulations included cases in which the b_i's were uniform ranging from 0.5 to 1.5 or had a gamma distribution formed by summing two exponentials each with a mean of 0.5. These simulations indicated that with random b_i's the sum diverges from the normal as n increases; for $n > 10$, the Geary test virtually always rejected normality. Thus, it appears that the sum in (30) converges to approximate normality only when the b_i's are (nearly) constant.

The models we have considered thus far have structures quite unlike the serial models commonly used in stage analysis of RT (e.g., Sternberg, 1969), because in stage models total RT is the sum of the processing times for individual stages. In principle, however, the lognormal may also arise from a serial process. Thorin (1977) has shown that the lognormal may be arbitrarily well approximated by the convolution of a finite number of gamma densities (cf. Feller, 1971, p. 47). In order to approximate the lognormal, the gamma densities must differ in their scale and shape parameters. Thus, a serial model could produce a lognormal distribution of RTs if the individual stages had an appropriate combination of gamma

distributions. In practice, however, it appears somewhat unlikely that an appropriate combination of gamma distributions would arise by chance, at least for serial models with small numbers of stages. Simulations were conducted with two, four, and eight stage models. First, the parameters of the gamma distribution for each stage were randomly selected. The shape parameter was selcted from a uniform distribution over the range of 1–10. It seems reasonable to allow at most an order of magnitude variation in the shape parameter, since different stages seem unlikely to differ more substantially than this in the processes determining their durations. The scale parameter was selected so that true mean stage durations would be distributed (across models) normally with a mean of 50 and a standard deviation of four. In the absence of other information, it seems reasonable to assume that stage durations are approximately equal. Second, 10,000 RTs were randomly generated from the model with the selected parameter values. Third, the null hypothesis of a normal distribution of ln(RT) was tested with an appropriately modified version of the Kolmogorov–Smirnov test (Dallal & Wilkinson, 1986; Lilliefors, 1967). The null hypothesis could be rejected ($\alpha = 0.05$) for 92.5%, 77.1%, and 43.5% of the parameter combinations selected for the two, four, and eight stage models, respectively. Thus, sequences of small numbers of stages having gamma durations with independently selected parameter values are unlikely to produce RT distributions for which the lognormal provides an acceptable fit. In additional simulations the shape parameter was selected from uniform distributions over the intervals 1–4 and 1–20. With the narrower range (1–4), power increased dramatically (above 93% in all cases), and with the wider range it decreased to approximately 87%, 61%, and 40% for two, four, and eight stage models, respectively. Still other simulations used standard deviations of one or ten, instead of four, but this variation had little effect on simulation outcomes.

3. Conclusions

In this paper we have identified several types of models that produce either exact or asymptotic lognormal response latency distributions, and it is evident that there are several theoretically meaningful ways in which lognormal latency distributions can arise. The most prominent ones involve either

1. exponentially transformed normal random variables
2. a product of independent random variables
3. or both (since the product of lognormally distributed is also lognormally distributed).

We have demonstrated that it is possible to construct reasonable RT models with these mathematical properties, although further research will be needed to determine the applicability of these models within particular experimental contexts. In any case, it is interesting to note that these models are conceptually very different

from the discrete-stage serial models involved in much of the RT-theorizing over the past 20 years (e.g., Sternberg, 1969), although certain serial models can also generate the lognormal. Thus, the lognormal distribution, which has been fitted successfully to empirical RT distributions by several researchers starting with Schlosberg and Heineman (1950), can now be regarded as having the status of a theoretically based distribution rather than a totally ad-hoc one in RT research.

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